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Abstract

A theorem from control theory relating the Lie algebra generated by vector fields on a manifold to the controllability of the dynamical system is shown to apply to holonomic quantum computation. Conditions for deriving the holonomy algebra are presented by taking covariant derivatives of the curvature associated with a non-Abelian gauge connection. When applied to the optical holonomic computer, these conditions determine that the holonomy group of the two-qubit interaction model contains $SU(2) \times SU(2)$. In particular, a universal two-qubit logic gate is attainable for this model.

PACS numbers: 03.67.Lx, 03.65.Vf

1. Introduction

Controlling quantum dynamics to effect a desired unitary evolution is a fundamental issue in quantum computation. Full control over the system dynamics and hence the ability to realize any logic gate is called *universal quantum computation*. In recent years, there has been considerable interest in conditions for universality and it has been proved that for an n-level quantum system universality is a *dense* condition, being satisfied by *almost all* computational models [1–3].

Despite the richness of the mathematical model, the effects of quantum noise make it a tremendous challenge to manifest this property in nature. Zanardi and Rasetti [4] have proposed a novel methodology for the control of quantum information which may provide a resolution to these competing phenomena. Holonomic quantum computation (HQC), as introduced in [4], is a theoretically appealing model that can provide universal computation, and, due to the geometrical nature of the framework, possesses intrinsic robustness against decoherence and control imperfection.

In this paper, we are primarily interested in the existence of the logic gates available to the experimentalist within the HQC framework. Specifically, we report a result from geometric control theory that simplifies the calculation of the holonomy group associated with a non-Abelian gauge connection. The application of this theorem to HQC establishes conditions for universality in general and in particular proves the universality of the *optical holonomic computer* [4–8]. We refer the reader to the literature for a more detailed exposition of the HQC set-up [7, 8], techniques for the calculation of the holonomies [8, 9], and its intrinsic fault-tolerance [10, 11].

The methodology is briefly described as follows. The quantum code is realized by the n-dimensional eigenspace, $\mathcal{C}$, of an n-fold degenerate Hamiltonian H_0 with eigenvalue E_0. Let M be a d-dimensional real parameter space. We suppose that the experimentalist can implement unitary transformations $\mathcal{U}(\eta)$ depending continuously on the parameter $\eta \in M$. A set of isospectral Hamiltonians is formed by the adjoint orbit of H_0

$$\mathcal{O}(H_0) \equiv \{\mathcal{U}(\eta)H_0\mathcal{U}^\dagger(\eta), \ \eta \in M\}. \tag{1}$$

Let $\gamma : [0, T] \to M$ be a closed curve in parameter space. If we traverse this loop sufficiently slowly, the adiabatic theorem ensures that no energy level crossing will occur and $\mathcal{O}(H_0)$ forms a family of Hamiltonians that drive the dynamics. Let $|\psi\rangle_i \in \mathcal{C}$ be the initial state of the system, then after completing an adiabatic loop in parameter space, the initial and final states are related by

$$|\psi\rangle_f = \mathrm{e}^{\mathrm{i}E_0 T}\Gamma_A(\gamma)|\psi\rangle_i \in \mathcal{C} \tag{2}$$

where $\mathrm{e}^{\mathrm{i}E_0 T}$ is the dynamical phase which will be omitted in the following by setting $E_0 = 0$. The matrix, $\Gamma_A(\gamma) \in U(n)$, is the *holonomy* associated with the loop γ and for $n > 1$, these non-Abelian holonomies are the logic gates of the quantum computer. The holonomy or *geometric phase* [12] depends only on the geometry of the loop γ and can be expressed as

$$\Gamma_A(\gamma) = \mathbf{P}\exp\oint_\gamma A \tag{3}$$

where $\mathbf{P}$ denotes the path ordering and the skew-symmetric matrix valued one-form $\mathcal{A}$ is known as the Wilczek–Zee connection [13] with matrix elements

$$A_{\eta_i}^{\bar{\varrho}\varrho} \equiv \langle\bar{\varrho}|\mathcal{U}^\dagger(\eta)\frac{\partial}{\partial\eta_i}\mathcal{U}(\eta)|\varrho\rangle. \tag{4}$$

In this way, Γ_A may be considered to be a map from the loop space of M to the matrix Lie group $U(n)$. The set $\mathrm{Hol}(A) := \{\Gamma_A(\gamma) \mid \gamma \in M\}$ forms a group under composition of loops in parameter space and is, in general, a *subgroup* of $U(n)$. $\mathrm{Hol}(A)$ is said to be the *holonomy group*, and the corresponding Lie algebra is known as the *holonomy algebra*. When $\mathrm{Hol}(A) = U(n)$, the connection A is *irreducible* [4]. Clearly, irreducibility of the connection is sufficient for universal computation since any n-level unitary transformation may be applied to the code $|\psi\rangle \in \mathcal{C}$.

2. Universality

Since the work of Montgomery [14–17], mathematicians and engineers have cast certain problems in the modelling and control of dynamical systems subject to non-holonomic constraints in the language of gauge theory. These constraints come in two main varieties. A cat in free fall experiences a *dynamical* constraint set by the requirement that its angular momentum remains constant throughout its descent. An upside down cat with initial angular momentum zero endeavours to achieve a rotation and land upright by altering its shape. When a

mechanical system interacts with its environment, *kinematic* constraints are often encountered. For example, kinematic constraints are in force for a mobile robot with two independently controlled rear wheels subject to a no-slip constraint against the rolling surface [18].

Mathematically, the scenario is described by a *connection on a principal G-bundle* [19]. We recall these constructions and provide a geometric setting for the remarks made in the introduction. A principal G-bundle is formed by manifolds Q (total space), M (base space), a free Lie group action $\Phi : G \times Q \to Q$ and the canonical projection $Q \overset{\pi}{\to} Q/G \equiv M$. If U is a neighbourhood of M, then Q is locally diffeomorphic to the product $U \times G$. A smooth map $\sigma : U \to Q$, such that $\pi \circ \sigma = \mathrm{id}_M$ is called a local section over U.

The fibre $\pi^{-1}(p)$ over a point $p \in M$ is identical to the group orbit and is denoted by G_p. Let $\mathfrak{g}$ denote the Lie algebra of G. For any element, $\xi \in \mathfrak{g}$ the group action $\Phi_{\exp(t\xi)}q$ defines a curve though $q \in Q$. The *infinitesimal generator* ξ_q of the group action is defined as the tangent vector

$$\xi_q := \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \Phi_{\exp(t\xi)} q. \tag{5}$$

The *vertical subspace* V_qQ is defined to be the subspace of T_qQ that is tangent to the fibre G_p, by the previous definition we have the identification $V_qQ \cong \mathfrak{g}$.

A connection $\mathcal{A}$ on Q is an Ad-equivariant Lie algebra-valued one-form $\mathcal{A} : TQ \to \mathfrak{g}$ such that $\mathcal{A}(\xi_q) = \xi$ for all $\xi \in \mathfrak{g}$. The horizontal space H_qQ is the linear space $H_qQ := \{X_q \in T_qQ \mid \mathcal{A}_q(X_q) = 0\}$. The local connection form A is defined with respect to a local section $A = \sigma^*(\mathcal{A})$. These definitions provide the splitting

$$T_qQ = V_qQ \oplus H_qQ \tag{6}$$

of the tangent vectors into horizontal and vertical components. Note that

$$V_qQ = \mathrm{Ker}\, T_q\pi \quad \text{and} \quad H_qQ = \mathrm{Ker}\, \mathcal{A}_q. \tag{7}$$

The projection map at a point defines an isomorphism from the horizontal space to the tangent space to the base space by $T_q\pi : H_q \to T_{\pi(q)}M$. Thus a curve $q(t) \in Q$ defines a curve in the base space by specifying a tangent vector at each point $\pi(q(t)) = p(t)$. The properties of the connection and the uniqueness theory of ODEs provide the reverse procedure of reconstructing a curve in the total space, given a curve in the base space called the *horizontal lift* [19]. Denote the horizontal lift of $X \in TM$ by X^h and the horizontal part of $Z \in TQ$ as hZ. The horizontal lift of any closed curve in the base space maps the fibre to itself and corresponds to a group element g by the automorphism $q_f = \Phi_g q_i$. Assuming direct control over the base velocities, we seek a closed curve in the base space that achieves a desired group translation in the fibre.

In the case of the cat (robot), the conservation law (no-slip constraint) defines a connection on a principal bundle with group action $SO(3) \times Q \to Q(SE(2) \times Q \to Q)$. The key point in the modelling and control of these systems is that the horizontal distribution, defined by the connection, encodes the constraint information. Thus a curve $q(t) \in Q$ satisfies the constraints if its tangent vector $X_{q(t)}$ lies in $H_{q(t)}Q$ for all t. Such a curve is called horizontal. Given an initial state q_i, the feasibility of reaching a final state q_f is then equivalent to the existence of a horizontal curve joining q_i and q_f. For control systems of this type, it is natural to define the *reachable set* from q_i as the set of points $q \in Q$ that lie on a horizontal curve originating at q_i.

Theorem (Chow)**.** *Suppose Q is connected. Let $X_i^h, 1 \leqslant i \leqslant d$ be a local frame of the horizontal space at q. Then any two points of Q can be joined by a horizontal curve if the*

iterated Lie brackets $\left[X_{i_k}^h, \left[X_{i_{k-1}}^h, \ldots, \left[X_{i_2}^h, X_{i_1}^h\right]\ldots\right] \right.$ *evaluated at* $q \in Q$ *span the tangent space* $T_q Q$ *for all* q.

For control systems without drift, this classical theorem [20] gives a sufficient condition to determine whether the reachable set is the entire manifold Q. If the Lie brackets defined in the theorem fail to span all of $T_q Q$, the reachable set may be characterized as follows. Denote the subspace of $T_q Q$ defined in the theorem as

$$\Delta_q = \mathrm{span}\left\{\left[X_{i_k}^h, \left[X_{i_{k-1}}^h, \ldots, \left[X_{i_2}^h, X_{i_1}^h\right]\ldots\right]\right]; 1 \leqslant i_k \leqslant d, 1 \leqslant k < \infty\right\}. \quad (8)$$

The sub-bundle $\Delta = \bigcup_q \Delta_q$ forms, by construction, an involutive distribution on the manifold Q. If the rank of Δ_q is constant as q is varied, the Frobenius theorem [21, 22] then asserts the existence of an *integral submanifold* $\tilde{Q} \subset Q$ with Δ as its tangent bundle. This submanifold is invariant under the constrained dynamics and forms the reachable set.

These arguments and the general principle embodied in the theorem are well known in the quantum computation literature. In the usual *dynamical* approach to quantum computing, the experimentalist has a repertoire of Hamiltonians $\{H_l\}_{l=1}^r$ that act on the quantum state. If the Lie algebra generated by the $-\mathrm{i}H_l$ under commutation is equal to $su(n)$ (or $u(n)$), then the system is deemed capable of performing universal computation. This is equivalent to the notion of *complete controllability* for quantum systems [23–25]. It is not surprising, then, that Chow's theorem is decisive in the holonomic framework as well.

To establish universality of HQC, we must show that the holonomy group is rich enough to generate a universal set of logic gates. The holonomy group is determined by the $\mathfrak{g}$-valued *curvature two-form* defined by

$$\mathcal{F}(X_1, X_2) = \mathbf{d}\,\mathcal{A}(hX_1, hX_2) \quad (9)$$

where $\mathbf{d}$ denotes exterior differentiation. To evaluate the curvature we employ the structure equation

$$\mathcal{F}(X_1, X_2) = \mathrm{d}\mathcal{A}(X_1, X_2) + [\mathcal{A}(X_1), \mathcal{A}(X_2)] \quad (10)$$

for $X_1, X_2 \in T_q Q$. Since the connection $\mathcal{A}$ evaluates to zero on horizontal vectors, the structure equation implies that $\mathcal{F}(X_1^h, X_2^h) = -\mathcal{A}([X_1^h, X_2^h])$. Recalling that a connection can be defined as a projection of a vector $X \in T_q Q$ onto the vertical subspace $V_q Q \cong \mathfrak{g}$, we see that $\mathcal{F}(X^h, Y^h)$ is the vertical component of the vector $[X_1^h, X_2^h]$. The Ambrose–Singer theorem expresses the holonomy group associated with the connection in terms of the curvature. As stated in [19]:

Theorem (Ambrose–Singer). *Let Q be a principal G-bundle over a manifold M. The Lie algebra $\mathfrak{h}$ of the holonomy group $\mathrm{Hol}_{q_0}(\mathcal{A})$ of a point $q_0 \in Q$ agrees with the subalgebra of $\mathfrak{g}$ spanned by the elements of the form $\mathcal{F}_q(X^h, Y^h)$ where $X^h, Y^h \in H_q Q$ and 'q is a point on the same horizontal lift as q_0'.*

This theorem has been quoted by other authors to provide sufficient conditions for the universality of HQC. The quoted statement, however, demands that we evaluate the curvature on the horizontal space at every point q that is reachable from q_0 via a horizontal curve. This set of points, however, is the reachable set as defined above.

More tractable conditions are obtained from Chow's theorem. By the above reasoning, elements of the form

$$\mathcal{F}(X_i^h, X_j^h) = -\mathcal{A}([X_i^h, X_j^h]) \qquad \mathcal{F}(X_i^h, X_k^h) = -\mathcal{A}([X_i^h, X_k^h]), \ldots \quad (11)$$

contribute a set of group directions obtained from brackets of horizontal vectors. According to Chow's theorem we must compute all the *iterated* Lie brackets of horizontal vectors.

The vertical component of the vector $\left[X_1^h, \left[X_2^h, X_3^h\right]\right]$ is given by $D_{X_1^h}\mathcal{F}\left(X_2^h, X_3^h\right)$ and higher order Lie brackets are expressed as higher order covariant derivatives of the curvature.

Corollary. *Suppose Q is connected. The holonomy algebra at a point $q_0 \in Q$ is spanned by the curvature forms $\mathcal{F}\left(X_{i_1}^h, X_{i_2}^h\right)$ and the covariant derivatives $D_{X_{i_k}^h} D_{X_{i_{k-1}}^h} \cdots D_{X_{i_3}^h} \mathcal{F}\left(X_{i_2}^h, X_{i_1}^h\right)$ evaluated at q_0.*

This result appears in Montgomery [14]. A proof can be found in [17]. (See also [26].) It is interesting to note that Montgomery's original motivation, in addition to the cat's problem, was the optimal control of spin systems.

To apply this result to HQC we identify the relevant manifolds and the horizontal direction. Following Fujii [6, 7], let $\mathcal{H}$ be a separable Hilbert space and define the manifolds

$$\mathrm{St}_n(\mathcal{H}) := \{V = (v_1, \ldots, v_n) \in \mathcal{H} \times \cdots \times \mathcal{H} \mid V^\dagger V = \mathrm{Id}_{n\times n}\}$$
$$\mathrm{Gr}_n(\mathcal{H}) := \{X \in B(\mathcal{H}) \mid X^2 = X, X^\dagger = X, \mathrm{tr}X = n\}$$

where $B(\mathcal{H})$ denotes the set of bounded linear operators on $\mathcal{H}$. These manifolds are known as the Stiefel and Grassmann manifolds, respectively. They form a principal bundle with the (right) $U(n)$ action on $\mathrm{St}_n(\mathcal{H})$ and the projection $\pi : \mathrm{St}_n(\mathcal{H}) \to \mathrm{Gr}_n(\mathcal{H})$ given by $\pi(V) = VV^\dagger$. Denote this $U(n)$-bundle by P. Let M be the parameter space and let the map $\Pi : M \to \mathrm{Gr}_n(\mathcal{H})$ be given. The principal bundle of interest is then formed by the pullback of P by Π, $Q = \Pi^*P$ with total space

$$Q = \{(\lambda, V) \in M \times \mathrm{St}_n(\mathcal{H}) \mid \Pi(\lambda) = \pi(V)\}$$

over the base manifold M. To be precise, the *left* action of the matrix acting on a vector $|\psi\rangle \in \mathbf{C}^n$ as defined by (2) takes place in the $\mathbf{C}^n$ vector bundle associated with Q [6, 7].

An important special case of this construction, known as the **CP**^{n} model, has been shown to be *generically* irreducible [4]. In this case, $\mathcal{H} \cong \mathbf{C}^{n+1} = \{|\alpha\rangle\}_{\alpha=1}^{n+1}$ and H_0 has an n-dimensional degenerate subspace. The parameter space $M = \mathbf{CP}^n$ is isomorphic to the orbit of H_0,

$$\mathcal{O}(H_0) \cong \frac{U(n+1)}{U(n) \times U(1)} \cong \frac{SU(n+1)}{U(n)} \cong \mathbf{CP}^n.$$

Thus Π is a surjective map of $M = \mathbf{CP}^n$ onto $\mathcal{O}(H_0) \cong \mathrm{Gr}_{1,n+1}$. Due to the large parameter space, this model can be shown to be irreducible by considering the span of the curvature form only (and not its covariant derivatives). Note that this model requires control over $2n = \dim_{\mathbf{R}}\mathbf{CP}^n$ parameters to control an n-level system.

In any case, the Wilczek–Zee connection (4) with its built-in Hermitian structure defines the horizontal subspace by identifying horizontal vectors as those which are orthogonal to the fibre [8, 13]. When applied to HQC, this result represents a significant reduction in the control resources necessary for universality and thus broadens the class of quantum evolutions that are capable of computation. Indeed, if one considers the span of the curvature form only, then one *incorrectly* concludes that a necessary condition for irreducibility of a connection associated to an n-level system is given by $d(d-1)/2 \geqslant n^2$ where $d = \dim_{\mathbf{R}}M$ [8].

3. Optical holonomic computer

It is widely believed that coherent superposition alone cannot account for the exponential speed-up sought in the realization of a quantum computer. Quantum entanglement must also be present [27]. As observed in [4, 8] the **CP**^{n} model does not possess the multi-partite structure

necessary for encoding entangled states. Attention is, therefore, directed to physical systems that have a multi-partite structure built in from the start *and* over which the experimentalist can exert control. While the results presented here apply to any HQC set-up, we are interested in a model coming from quantum optics where displacing and squeezing devices realize control operations acting on laser beams in a non-linear Kerr medium [4–8].

Let $a^\dagger, a$ be the creation and annihilation operators of the harmonic oscillator and let $n = a^\dagger a$ be the number operator. Let $\mathcal{H}$ be the Fock space generated by $a^\dagger$ and a with basis $\{|\nu\rangle : \nu = 0, 1, \ldots\}$. Each qubit is encoded in the degenerate subspace of the interaction Hamiltonian

$$H^1 = X\hbar n(n-1) \tag{12}$$

where X is a constant [5]. This computing scheme scales to a system of m qubits by employing m lasers to form the product basis $|\nu_1 \nu_2 \ldots \nu_m\rangle = |\nu_1\rangle \otimes |\nu_2\rangle \otimes \ldots |\nu_m\rangle$ where $\nu_i \in \{0, 1\}$. In accordance with the quantum circuit model, a control strategy is devised to implement all single qubit rotations and a non-trivial two-qubit transformation. A fundamental result then asserts that universality of the entire quantum register of m qubits can be achieved by this set of two-level local transformations [28].

Single-mode squeezing and displacing operators are employed to control the single qubit,

$$S(\mu) = \exp(\mu a^{\dagger 2} - \bar{\mu} a^2) \qquad D(\lambda) = \exp(\lambda a^\dagger - \bar{\lambda} a) \tag{13}$$

where $\mu, \lambda \in \mathbf{C}$. These operators define a two-parameter orbit of H^1 under the unitary transformation $\mathcal{U}(\lambda, \mu) = D(\lambda)S(\mu)$,

$$\mathcal{O}(H^1) = \mathcal{U}(\lambda, \mu)H^1\mathcal{U}^\dagger(\lambda, \mu). \tag{14}$$

The holonomy group associated with loops in the (λ, μ) parameter space is $U(2)$ [5, 6].

To prove universality of the computational model it suffices to generate non-trivial $U(4)$ holonomies. For the two-qubit system, the Hamiltonian is given by

$$H^{12} = X\hbar n_1(n_1 - 1) + X\hbar n_2(n_2 - 1) \tag{15}$$

where n_i is the number operator for the ith beam. Two-mode squeezing and displacing operators realize control operations,

$$M(\zeta) = \exp(\zeta a_1^\dagger a_2^\dagger - \bar{\zeta} a_1 a_2) \qquad N(\xi) = \exp(\xi a_1^\dagger a_2 - \bar{\xi} a_1 a_2^\dagger) \tag{16}$$

where $\zeta = r_2 \mathrm{e}^{i\theta_2}, \xi = r_3 \mathrm{e}^{i\theta_3} \in \mathbf{C}$. In the adiabatic limit, the adjoint orbit under the action $\mathcal{U}(\xi, \zeta) = N(\xi)M(\zeta)$,

$$\mathcal{O}(H^{12}) = \mathcal{U}(\xi, \zeta)H^{12}\mathcal{U}^\dagger(\xi, \zeta) \tag{17}$$

drives the dynamics. The degenerate subspace of H^{12} is given by the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Set $|\mathrm{vac}\rangle = (|00\rangle, |01\rangle, |10\rangle, |11\rangle) \in \mathrm{St}_4(\mathcal{H} \otimes \mathcal{H})$.

We characterize the reachable set from $q_0 = (m, |\mathrm{vac}\rangle) \in Q$ by applying the conditions obtained from Chow's theorem. The local connection coefficients A_ν are written in terms of the base variables only, $(r_2, \theta_2, r_3, \theta_3) \in M \subset \mathbf{C}^2$ [5, 8],

$$A_{r_2} = \begin{bmatrix} 0 & 0 & 0 & -\mathrm{e}^{-i\theta_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{e}^{i\theta_2} & 0 & 0 & 0 \end{bmatrix} \qquad A_{r_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{e}^{-i\theta_3} & 0 \\ 0 & \mathrm{e}^{i\theta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \left(2\cosh^2 r_2 - 1\right)$$

$$A_{\theta_2} = \begin{bmatrix} 0 & 0 & 0 & \mathrm{e}^{-i\theta_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{e}^{i\theta_2} & 0 & 0 & 0 \end{bmatrix} \frac{i}{2}\sinh 2r_2 + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \frac{i}{2}(\cosh 2r_2 - 1)$$

$$A_{\theta_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-i\theta_3} & 0 \\ 0 & e^{i\theta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{i}{2} \cosh 2r_2 \sin 2r_3 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} i \sin^2 r_3.$$

The nonzero local curvature forms $F_{\mu\nu}$,

$$F_{r_2 r_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -e^{-i\theta_3} & 0 \\ 0 & e^{i\theta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} 2\sinh 2r_2 \qquad F_{r_2\theta_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} 2i \sinh 2r_2$$

$$F_{r_2\theta_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-i\theta_3} & 0 \\ 0 & e^{i\theta_3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} i \sin 2r_3 \sinh 2r_2 \qquad F_{r_3\theta_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} i \sin 2r_3 \sinh^2 2r_2$$

span $su(2) \times u(1)$.

The block structure of these matrices suggests that new group directions may be obtained by taking covariant derivatives of $F_{r_2\theta_2}$ along the base coordinate vectors $\frac{\partial}{\partial\theta_2}$ and $\frac{\partial}{\partial r_2}$,

$$D_{\frac{\partial}{\partial\theta_2}} F_{r_2\theta_2} = \begin{bmatrix} 0 & 0 & 0 & -e^{-i\theta_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{i\theta_2} & 0 & 0 & 0 \end{bmatrix} 2\sinh^2 2r_2$$

$$D_{\frac{\partial}{\partial r_2}} F_{r_2\theta_2} = \begin{bmatrix} 0 & 0 & 0 & e^{-i\theta_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{i\theta_2} & 0 & 0 & 0 \end{bmatrix} -4i \sinh 2r_2 + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} 4i \cosh 2r_2$$

$$D_{\frac{\partial}{\partial\theta_2}} D_{\frac{\partial}{\partial\theta_2}} F_{r_2\theta_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} 2i \sinh^3 r_2 + \begin{bmatrix} 0 & 0 & 0 & e^{-i\theta_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e^{i\theta_2} & 0 & 0 & 0 \end{bmatrix} 2i \sinh^2 2r_2 \cosh 2r_2.$$

These matrices and the independent contributions of the previous set span the Lie algebra $su(2) \times su(2) \times u(1) \subset u(4)$. The connection is not irreducible; however, the product structure of the subgroup shows that non-trivial $U(4)$ transformations are attainable. This result reconciles conflicting results from the literature. One concludes after consideration of the curvature form only the holonomy group to be $SU(2) \times U(1)$ [7]. However, a variant of the *square root of SWAP* gate

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix} \tag{18}$$

can been explicitly constructed [8]. This universal transformation is an element of $SU(2) \times SU(2)$.

4. Concluding remarks

Holonomic quantum computation represents a novel approach to quantum computing by employing non-Abelian geometric phases to perform information processing. The geometric phase is a beautiful phenomenon with a long history in physics, mathematics and engineering. We have shown that a result from control theory provides key insight into the foundations of HQC and opens up the possibility that more physical systems will be amenable to this approach. For a particular manifestation of HQC—the optical holonomic computer—the results presented here provide new understanding of the controlled interactions attainable in the experimental set-up. Moreover, we have proved the existence of a universal set of logic gates.

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